Hydrodynamic singular regimes in 1+1 kinetic models and spectral numerical methods

Laurent Gosse, Nicolas Vauchelet

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Laurent Gosse *

IAC–CNR "Mauro Picone" (sezione di Roma)
Via dei Taurini, 19 - 00185 Rome, Italy

Nicolas Vauchelet

Sorbonne Universités, UPMC Univ Paris 06, Inria, Lab. J.L. Lions UMR 7598
CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

Abstract

Classical results from spectral theory of stationary linear kinetic equations are applied to efficiently approximate two physically relevant weakly nonlinear kinetic models: a model of chemotaxis involving a biased velocity-redistribution integral term, and a Vlasov-Fokker-Planck (VFP) system. Both are coupled to an attractive elliptic equation producing corresponding mean-field potentials. Spectral decompositions of stationary kinetic distributions are recalled, based on a variation of Case's elementary solutions (for the first model) and on a Sturm-Liouville eigenvalue problem (for the second one). Well-balanced Godunov schemes with strong stability properties are deduced. Moreover, in the stiff hydrodynamical scaling, an hybridized algorithm is set up, for which asymptotic-preserving properties can be established under mild restrictions on the computational grid. Several numerical validations are displayed, including the consistency of the VFP model with Burgers-Hopf dynamics on the velocity field after blowup of the macroscopic density into Dirac masses.

Key words: Chemotaxis modeling; discrete velocity kinetic model; non-conservative products; Vlasov-Poisson Fokker-Planck equation; Asymptotic-Preserving and Well-Balanced scheme.
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* Corresponding Author

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1 Introduction and theoretical setup

This paper is devoted to the numerical discretization of kinetic systems at hydrodynamic scaling in the strongly attractive setting. We are mainly interested in two systems, one modeling chemotaxis, and the Vlasov-Poisson Fokker-Planck (VPFP) system in plasma physics. Our aim is to propose accurate numerical schemes able to cope with hydrodynamic limit i.e. consistant with the limiting macroscopic model for which solutions may blowup into Dirac deltas in finite time.

1.1 Presentation of the two main kinetic models

The present paper, a followup of [36,32], is devoted to the following two models:

- a first one describing the motion by chemotaxis, proposed in e.g. [41,17,26] :

$$\partial_t f + v\partial_x f = \frac{1}{\varepsilon}\left(h(v\partial_x S)\rho - f(t,x,v)\right), \quad 0 \le h \le 1, \quad \int_{-1}^{1} h(v\partial_x S)dv = 1, \tag{1}$$

$$-\partial_{xx} S + S = \rho := \int_{-1}^{1} f(t,x,v')dv'. \tag{2}$$

In this model, the distribution function f governs the dynamics of bacteria or cells moving by chemotaxis, i.e. responding to a gradient of a chemical S generating by cells themselves. In this equation $x \in \mathbb{R}$, $v \in [-1,1]$ and $t > 0$. In order to guarantee the condition on h, we may assume

$$h(u) = \frac{1}{2} + h_1(u), \quad \text{with} \quad h_1(-u) = -h_1(u), \quad -\frac{1}{2} \le h_1(\cdot) \le \frac{1}{2}. \tag{3}$$

When taxis dominates the unbiased movements of cells, $\varepsilon \ll 1$, so the dynamics is strongly aggregative. Then, the first moments in v of the solution to the mesoscopic model can be approximated by a simplified macroscopic equation, of aggregation type [21,35]. Indeed, taking the two first moments of (1), we have

$$\partial_t \rho + \partial_x J = 0, \qquad \varepsilon(\partial_t J + \partial_x q) = a(\partial_x S)\rho - J, \tag{4}$$

where it was set

$$J = \int_{-1}^{1} v f(v)dv, \qquad q = \int_{-1}^{1} v^2 f(v)dv, \qquad a(u) = \int_{-1}^{1} v h(vu)dv = \int_{-1}^{1} v h_1(vu)dx, \tag{5}$$

and (3) was used to produce the last equality. We deduce easily (at least formally) from (4), that at the limit $\varepsilon \to 0$, an aggregation equation emerges,

$$\boxed{\partial_t \rho + \partial_x(a(\partial_x S)\rho) = 0, \qquad -\partial_{xx} S + S = \rho.} \tag{6}$$

- A second model of interest in this work is the Vlasov-Poisson-Fokker-Planck (VPFP) system. It governs the dynamics of the distribution function f of charge carriers within an electric field $E = \partial_x \phi$ generated by the charge carriers themselves. The system reads [40] :

$$\varepsilon(\partial_t f + v\partial_x f) + \partial_x \phi \cdot \partial_v f = \frac{1}{\theta}\partial_v(vf + \kappa\partial_v f), \tag{7}$$

$$-\partial_{xx}\phi = \rho := \int_{\mathbb{R}} f(v)dv. \tag{8}$$

The high field regime corresponds to letting $\varepsilon \to 0$. As above, we take the two first moments of (7),

$$\partial_t \rho + \partial_x J = 0, \qquad \varepsilon\theta(\partial_t J + \partial_x q) = \theta \partial_x \phi \rho - J, \tag{9}$$

where we set $J = \int_{\mathbb{R}} v f(v) dv$, and $q = \int_{\mathbb{R}} v^2 f(v) dv$. We deduce easily at least formally from (9), that at the limit $\varepsilon \to 0$ we recover the aggregation equation

$$\boxed{\partial_t \rho + \theta \partial_x(\partial_x \phi \rho) = 0, \qquad -\partial_{xx}\phi = \rho.} \tag{10}$$

The above formal computation of the hydrodynamical limit has been done rigorously in [40,35]. We aim at providing numerical counterpart of this result by introducing an asymptotic preserving numerical scheme. The aggregation equation has been studied by many authors, for instance [5,13,35,37,15]. It is well-known that weak solutions of the aggregation equation in the strongly attractive setting may blowup in finite time. Then numerical simulations of such system is a challenging issue. In [36,15], based on an analysis of weak measure solutions, numerical schemes allowing to recover the dynamics after blowup are provided. Numerical simulations of the aggregation equation before the blowup time are investigated in [14,18,12]. Numerical schemes for Vlasov-Fokker-Planck models, possibly coupled to attractive or repulsive Poisson equations, were studied in *e.g.* [52,53,38,19]; however, no mention of the transient spectral method we intend to present hereafter appeared, and no comparison with the limiting Burgers equation in the high field limit too, up to our knowledge.

1.2 Theoretical framework and important results

Useful theoretical tools and results are yet to be recalled; the interested reader may consult [6,35,37] for details. Hereafter $\mathcal{M}_b(\mathbb{R})$ refers to the linear space of local Borel measures on $\mathbb{R}$ with finite total mass; it is always endowed with its weak topology $\sigma(\mathcal{M}_b, C_0)$ where C_0 is the set of continuous functions vanishing at infinity. As usual, $\mathcal{P}_1(\mathbb{R})$ stands for the set of measures μ in $\mathcal{M}_b(\mathbb{R})$ admitting a bounded moment of order one, i.e. for which $\int_{\mathbb{R}} |x|\mu(dx) < \infty$. In [35,37], weak solutions of one-dimensional aggregation equations of the kind,

$$\partial_t \rho + \partial_x(a(W' * \rho)\rho) = 0, \qquad -\frac{d^2}{dx^2} W = \delta_0 + w, \tag{11}$$

are defined. In (11), δ_0 is the Dirac delta in 0 and w denotes a C^1 function. Such potential W belongs to the class of 'pointy potential'; for instance, the Morse potential, $W(x) = \frac{1}{2}e^{-|x|}$, or the Newtonian one, $W(x) = -\frac{1}{2}|x|$. In this approach, (11) is seen as a continuity equation with a velocity $a(W' * \rho)$ which depends nonlinearly on the solution. Solutions of such conservation laws may blow up in finite time (see e.g. [5]), so that Dirac masses may possibly appear in finite time; in such a case, the product $a(W' * \rho)\rho$ is ambiguous. A convenient way to deal with such a difficulty relies on the notion of **duality solutions**, introduced by Bouchut and James in [6] for 1D transport equations (see also [43,44]). Such solutions are weak, the test functions being so–called "reversible solutions" to the (dual) backward linear transport problem (see [6, §4.2] or [31, §2.2]).

Definition 1.1 *The set $(\rho, S) \in C(0,T;\mathcal{M}_b(\mathbb{R})) \times C(0,T;W^{1,\infty}(\mathbb{R}))$ is a* **duality solution** *to (6) if there exists $\widehat{a}_\rho \in L^\infty((0,T)\times\mathbb{R})$ and $\alpha(t) \in L^1_{loc}(0,T)$ such that, for $0 < t_1 < t_2 < T$,*

(1) ρ is a duality solution on (t_1, t_2) of the continuity equation,

$$\partial_t \rho + \partial_x(\widehat{a}_\rho \rho) = 0.$$

(2) $\forall t \in (0,T)$, $\widehat{a}_\rho = a(\partial_x S)$ almost everywhere (with respect to the Lebesgue measure) and $\partial_x(a(\partial_x S))(t,\cdot) \leq \alpha(t)$ in the sense of distributions (one-sided Lipschitz estimate).

(3) The elliptic equation for S holds in a weak sense: for any $\xi \in C^1(\mathbb{R})$,

$$\forall t \in (0,T), \qquad \int_{\mathbb{R}} \partial_x S(t,x) \cdot \partial_x \xi(x) + S(t,x) \cdot \xi(x)\, dx = \int_{\mathbb{R}} \rho(t,x)\xi(x)\, dx.$$

Replacing S by ϕ and taking $a = \theta \cdot Id$, we define by the same manner duality solutions for the continuity system (10). Stability of such duality solutions (see [6, Theorem 4.3.2]) allows to obtain existence of duality solutions to (11). Uniqueness relies on a definition of the nonconservative product, $a(\partial_x S)\rho$ in (6), $\rho.\partial_x\phi$ in (10), thanks to the Vol'pert calculus (or chain rule for BV function). More precisely, denoting A an antiderivative of a, Vol'pert calculus yields,

$$\partial_x(A(\partial_x S)) = \widehat{a}_\rho \partial_{xx} S, \qquad \frac{1}{2}\partial_x(\phi^2) = \widehat{E}_\rho \partial_{xx}\phi,$$

where $\widehat{a}_\rho = a(\partial_x S)$ a.e.[1] and $\widehat{E}_\rho = \partial_x\phi$ a.e.[1]. Accordingly,

Theorem 1.2 *[35, Theorem 3.9 and 3.10] Let $0 \le \rho^0 \in \mathcal{P}_1(\mathbb{R})$ and $T > 0$ be given:*

(1) for any nondecreasing function $a \in C^1 \cap L^\infty(\mathbb{R})$, there exists a unique duality solution (ρ, S) of (6) that satisfies in the distributional sense

$$\partial_t \rho + \partial_x J = 0, \qquad J = -\partial_x(A(\partial_x S)) + a(\partial_x S)\rho. \tag{12}$$

The macroscopic density of $(f_\varepsilon, S_\varepsilon)$ satisfying (1)–(2) weakly converges as $\varepsilon \to 0$ towards this unique duality solution in the sense of Def. 1.1: $\rho_\varepsilon(t,x) = \int_{\mathbb{R}} f_\varepsilon(t,x,v)\,dv \rightharpoonup \rho$ in $C([0,T], \mathcal{M}_b(\mathbb{R}) - \sigma(\mathcal{M}_b(\mathbb{R}), C_0))$.

(2) There exists a unique duality solution (ρ, ϕ) of (10) that satisfies in the distributional sense

$$\partial_t \rho + \partial_x J = 0, \qquad J = -\frac{\theta}{2}\partial_x(\partial_x\phi^2). \tag{13}$$

Again, its macroscopic density $(\rho_\varepsilon, \phi_\varepsilon)$ from (7)–(8) weakly converges as $\varepsilon \to 0$ toward this unique duality solution in the sense of Def. 1.1: $\int_{\mathbb{R}} f_\varepsilon(v)\,dv \rightharpoonup \rho$ in $C([0,T], \mathcal{M}_b(\mathbb{R}) - \sigma(\mathcal{M}_b(\mathbb{R}), C_0))$.

A byproduct of the definition of the macroscopic flux is that $\partial_x\phi$ satisfies a Burgers-Hopf equation $\partial_t u + \frac{\theta}{2}\partial_x u^2 = 0$, [40]. As $\rho = -\partial_{xx}\phi$ should be nonnegative, $\partial_x\phi$ is a non-increasing entropy solution of that conservation law: it matches its entropy solution. We aim at providing a numerical counterpart of Theorem 1.2, see Theorem 5.5.

1.3 Previous results for a two-stream model of Chemotaxis

For the sake of readability of the present paper, we recall briefly a previous work in a simpler framework. A particularly favorable case of (1)–(2) is the so-called "two-stream approximation", where the set of microscopic velocities shrinks to $\{-1, 1\}$: see [25,29,32] and [27, Chap. 8]. The Cattaneo or Greenberg-Alt model [33,25] describes the dynamics of the two distribution functions $f^\pm(t,x)$ of bacteria moving respectively in positive and negative directions, at time $t > 0$ and position $x \in \mathbb{R}$:

$$\partial_t f^\pm \pm \partial_x f^\pm = \pm\frac{1}{\varepsilon}\left([\frac{1}{2} + \phi(\partial_x S)]f^- - [\frac{1}{2} - \phi(\partial_x S)]f^+\right), \qquad x \in \mathbb{R}, \tag{14}$$

with initial/decay conditions $f^\pm(t=0,\cdot) = f_0^\pm$, $\lim_{x\to\pm\infty} f^\pm(t,x) = 0$, (weakly) coupled with (2). Macroscopic cell density is denoted $\rho = f^+ + f^-$; *turning kernels* $T^\pm := \frac{1}{2} \pm \phi(\partial_x S)$, see (3), correspond to bacteria reorientation rates, so $0 \le T^\pm \le 1$. Let $M = \|\rho\|_{L^1}$ the total (time-independent) mass, the latter condition on the turning kernel is satisfied provided a sub-characteristic condition is met:

$$\sup_{v\in[-M/2,M/2]} |\phi(v)| \le \frac{1}{2}. \tag{15}$$

[1] Here and in the following "almost everywhere" holds always with respect to the Lebesgue measure.

Defining $J = f^+ - f^-$, (14) rewrites as a semi-linear relaxation system,

$$\partial_t \rho + \partial_x J = 0, \qquad \partial_t J + \partial_x \rho = \frac{1}{\varepsilon}\big(2\phi(\partial_x S)\rho - J\big). \tag{16}$$

According to [27], a convenient manner of treating a weakly nonlinear model like (14)–(2) consists in building a Godunov scheme based on an exact Riemann solver for the slightly modified problem,

$$\partial_t f^\pm \pm \partial_x f^\pm = \pm \sum_{j\in\mathbb{Z}} \frac{\Delta x}{2\varepsilon}\left([1 + a^n_{j-\frac12}]f^- - [1 - a^n_{j-\frac12}]f^+\right)\cdot \delta(x - x_{j-\frac12}), \tag{17}$$

where $a^n_{j-\frac12}$ is a (yet supposedly known) "consistent discretization" of $2\phi(\partial_x S)(t^n, x_{j-\frac12})$. Across static discontinuities linked to Dirac masses, the unknowns $\bar f^\pm(x)$ jump according to stationary integral curves of (14),

$$\forall j, n \in \mathbb{Z}\times\mathbb{N}, \qquad 2\varepsilon\partial_x \bar f^\pm(x) = [1 + a^n_{j-\frac12}]\bar f^- - [1 - a^n_{j-\frac12}]\bar f^+,\ \bar f^+(0) = f^+_{j-1,n},\ \bar f^-(\Delta x) = f^-_{j,n}, \tag{18}$$

which are equivalent (there are the same quantity of microscopic and macroscopic variables) to,

$$\partial_x \bar J = 0, \qquad \varepsilon\partial_x \bar\rho(x) = a^n_{j-\frac12}\bar\rho - \bar J, \qquad x \in (0, \Delta x). \tag{19}$$

Tumbling terms are handled as a countable collection of "local scattering centers" [23], each of which inducing a 2×2 scattering matrix, $\mathcal{S}^n_{j-\frac12}$ relating incoming states (or "inflow data" of (18)) to outgoing ones:

$$\begin{pmatrix} f^+_{j-\frac12,n} \\ f^-_{j-\frac12,n}\end{pmatrix} := \begin{pmatrix} \bar f^+(\Delta x) \\ \bar f^-(0)\end{pmatrix} = \mathcal{S}^n_{j-\frac12}\begin{pmatrix} f^+_{j-1,n} \\ f^-_{j,n}\end{pmatrix}. \tag{20}$$

The resulting two-stream Godunov scheme,

$$f^+_{j,n+1} = f^+_{j,n} - \frac{\Delta t}{\Delta x}(f^+_{j,n} - f^+_{j-\frac12,n}), \qquad f^-_{j,n+1} = f^-_{j,n} + \frac{\Delta t}{\Delta x}(f^-_{j+\frac12,n} - f^-_{j,n}),$$

rewrites, in a vectorial form, (see further equations (42)–(44))

$$\begin{pmatrix} f^+_{j,n+1} \\ f^-_{j-1,n+1}\end{pmatrix} = (1 - \frac{\Delta t}{\Delta x})\begin{pmatrix} f^+_{j,n} \\ f^-_{j-1,n}\end{pmatrix} + \frac{\Delta t}{\Delta x}\mathcal{S}^n_{j-\frac12}\begin{pmatrix} f^+_{j-1,n} \\ f^-_{j,n}\end{pmatrix}. \tag{21}$$

Lemma 1.3 *(Lemma 4.1, [32]) For any $j, n \in \mathbb{Z}\times\mathbb{N}$, the scattering matrix for (14), $\mathcal{S}^n_{j-1/2}$, has the expression,*

$$\mathcal{S}^n_{j-1/2} = \begin{pmatrix} \dfrac{2c_L}{1 - c_R + c_L} & \dfrac{1 + c_R + c_L}{1 - c_R + c_L} \\ \dfrac{1 - c_R - c_L}{1 - c_R + c_L} & \dfrac{-2c_R}{1 - c_R + c_L}\end{pmatrix}, \tag{22}$$

with the signed coefficients, ($a^n_{j-\frac12,L}$ and $a^n_{j-\frac12,R}$ may be equal) :

$$c_R = \frac{a^n_{j-\frac12,R}}{1 - \exp(a^n_{j-\frac12,R}\Delta x/\varepsilon)} \le 0, \qquad c_L = \frac{a^n_{j-\frac12,L}}{1 - \exp(-a^n_{j-\frac12,L}\Delta x/\varepsilon)} \ge 0. \tag{23}$$

Further, under the discrete sub-characteristic condition,

$$\sup_{j,n\in\mathbb{Z}\times\mathbb{N}} |a^n_{j-\frac12,L/R}| \leq 1, \tag{24}$$

the matrix $\mathcal{S}^n_{j-1/2}$ is left-stochastic, so that (21) preserves both non-negativity and the total mass.

When put altogether, all values ρ_L and ρ_R at each interface induce sequences $(\rho^n_{j-\frac12,L})_j$ and $(\rho^n_{j-\frac12,R})_j$,

$$\forall j\in\mathbb{Z}, \qquad \rho^n_{j-\frac12,L} = f^+_{j-1} + f^-_{j-\frac12}\,, \qquad \rho^n_{j-\frac12,R} = f^+_{j-\frac12} + f^-_j\,.$$

Then according to the expression of $\mathcal{S}^n_{j-1/2}$ given in Lemma 1.3,

$$\rho^n_{j-\frac12,L} = \frac{2(1-c_R)}{1-c_R+c_L} f^+_{j-1} - \frac{2c_R}{1-c_R+c_L} f^-_j\,, \qquad \rho^n_{j-\frac12,R} = \frac{2c_L}{1-c_R+c_L} f^+_{j-1} + \frac{2(1+c_L)}{1-c_R+c_L} f^-_j\,. \tag{25}$$

These densities induce corresponding potentials S_L and S_R solving each elliptic problem, $-\partial_{xx}S_L+S_L=\rho_L$ and $-\partial_{xx}S_R+S_R=\rho_R$. Standard centered finite differences furnish approximations $S_{j-\frac12,L}$ and $S_{j-\frac12,R}$, respectively,

$$-\frac{S^n_{j+\frac12,L/R} - 2S^n_{j-\frac12,L/R} + S^n_{j-\frac32,L/R}}{\Delta x^2} + S^n_{j-\frac12,L/R} = \rho^n_{j-\frac12,L/R}\,, \tag{26}$$

Centered discretizations of partial derivatives of $S_{L/R}$ read

$$\partial_x S^n_{j,L} = \frac{S^n_{j+\frac12,L} - S^n_{j-\frac12,L}}{\Delta x} \qquad \text{and} \qquad \partial_x S^n_{j,R} = \frac{S^n_{j+\frac12,R} - S^n_{j-\frac12,R}}{\Delta x}\,.$$

Left and right values of the macroscopic velocity at interfaces $x_{j-\frac12}$ are computed by Vol'pert discretization :

$$\boxed{A'(\cdot)=a(\cdot), \qquad a^n_{j-\frac12,L/R} = \begin{cases} \dfrac{A(\partial_x S^n_{j,L/R}) - A(\partial_x S^n_{j-1,L/R})}{\partial_x S^n_{j,L/R} - \partial_x S^n_{j-1,L/R}}, & \text{if} \quad \partial_x S^n_{j,L/R} \neq \partial_x S^n_{j-1,L/R}\,; \\ 0, & \text{else}\,. \end{cases}} \tag{27}$$

According to (15), macroscopic velocities satisfy sub-characteristic conditions (24), as given in Lemma 1.3. In practice, computing $a^n_{j-\frac12,L/R}$ consists in iteratively solving a nonlinear system for $(S^n_{j-\frac12,L})_{j\in\mathbb{Z}}$ and $(S^n_{j-\frac12,R})_{j\in\mathbb{Z}}$: namely, (23)-(25)-(26)-(27). Our aim is to extend such algorithms to the more complex cases (1) and (7).

1.4 Outline of the paper

The manuscript is organized in a similar way as [32], the salient difference being that the original algorithms we both derive and study hereafter are not restricted to a two-stream approximation of each kinetic model, (1)–(2), or (7)–(8). As we deal with arbitrary discrete-ordinate approximations in the v variable (usually a Gauss quadrature rule on a bounded interval), computations are more involved and rely on elaborate theoretical grounds. As we intend to produce well-balanced approximations (in §3), at least in the kinetic regime $\varepsilon\simeq 1$, we mostly rely on discretizations based on "scattering matrices" which purpose is to handle, at each interface of the computational grid, the collision process. Expressions of these matrices follow from the resolution of stationary boundary-value problems (BVP) for each kinetic model: such results are recalled in §2. In particular, as h is not an even function, results of [1] do not suffice, so we shall rely on [42] for the chemotaxis model, and [9] (in the form of [27, Ch. 12]) for VFP. In order to produce reliable numerical schemes from spectral expansions, one may

need "matrix balancing" [45–47] or preconditioning techniques: this is presented in §3.2 and §3.3, respectively, and before a summary in §3.4. Corresponding numerical results are displayed in §4; in particular, a well-balanced benchmark, taken from [52], involves an exact solution of a repulsive Vlasov-Poisson Fokker-Planck (VPFP) equation (specific error estimates on simpler models are given in [2]). Consistency in the singular hydrodynamic limits (so–called Asymptotic-Preserving property) by means of an hybridized algorithm [29] is tackled in §5. As blowup of macroscopic densities into Dirac masses may happen in finite time, a convenient tool in order to ensure that correct dynamics will be reproduced at the numerical level is the Vol'Pert calculus [51]: see §5.1. Previous well-balanced Godunov schemes are amended accordingly to handle stiff and singular regimes in §5.2. Their consistency with the asymptotic behavior (given by aggregation equations of the type (6) or (10)) is studied in §5.3. Finally, some convergence results are given in §5.4, mostly under the assumption that convenient uniform bounds hold on sequences of numerical approximations. Finally, after a summary given in §5.5, new numerical results are displayed in §6; in particular, the agreement between Burgers-Hopf equation (10) for $E(t,x)$ and the hydrodynamic limit after blowup for (7)–(8) is observed numerically in §6.2, following [40] (see also (12) and (13)).

2 Spectral decomposition of stationary kinetic problems

Let us first consider that potentials (S in (1), ϕ in (7)) are given and so as to focus on corresponding solutions of the stationary problems (kinetic equations decouples from its elliptic counterpart). More precisely, let $\mathcal{L}$ be the collision operator on the right hand side in each kinetic model; for (1), $\mathcal{L}(f) = h(v\partial_x S)\int_{-1}^{1} f(v)\,dv - f$, for the VPFP system, we have $\mathcal{L}(f) = \frac{1}{\theta}\partial_v(vf + \kappa\partial_v f - \partial_x\phi f)$, so that:

$$\partial_t f + v\partial_x f = \frac{\mathcal{L}(f)}{\varepsilon}. \tag{28}$$

The operator $\mathcal{L}$ is linear, so the stationary problem reduces to

$$v\partial_x \bar{f} = \frac{\mathcal{L}(\bar{f})}{\varepsilon}. \tag{29}$$

A change of variable $x' = x/\varepsilon$ implies that we can pick $\varepsilon = 1$ without any loss of generality.

2.1 Chemotaxis as a biased velocity-redistribution model

The stationary problem corresponding to (1), when the chemoattractant concentration S is given leads to an equation like,

$$v\partial_x \bar{f} = \bar{h}(v)\int_{-1}^{1} \bar{f}(x,v')dv' - \bar{f}(x,v), \qquad x \in \mathbb{R}, v \in [-1,1], \tag{30}$$

where $\bar{h}$ is such as in (3). That problem being linear, a normalization of $f(x,v)$ is necessary: seeking a solution with separated variables, $\bar{f}(x,v) = X(v)Y(x)$, where $\int_{-1}^{1} X(v)dv = 1$, from (30), one reaches

$$vY'(x)X(v) = \bar{h}(v)Y(x) - X(v)Y(x), \qquad \frac{Y'(x)}{Y(x)} = \frac{\bar{h}(v)}{vX(v)} - \frac{1}{v}.$$

As this equality should hold for any value of x, v, there must be constants such that $Y'(x) = -\lambda Y(x)$ and $(1-\lambda v)X(v) = \bar{h}(v)$. Following e.g. [42], we denote $\bar{f}(x,v) = \exp(-\lambda x)\zeta_\lambda(v)$. Accordingly,

$$\int_{-1}^{1} \zeta_\lambda(v)\,dv = 1, \qquad \zeta_\lambda(v) = \frac{\bar{h}(v)}{1-\lambda v}. \tag{31}$$

The so–called dispersion relation of proper modes comes from the normalization condition,

$$0 = 1 - \int_{-1}^{1} \zeta_\lambda(v)\, dv = 1 - \int_{-1}^{1} \frac{\bar{h}(v)}{1-\lambda v}\, dv, \tag{32}$$

with an obvious root:

$$\lambda = 0, \qquad \zeta_0(v) = \bar{h}(v).$$

This is the only discrete eigenmode if $\bar{h}(v) \neq \bar{h}(-v)$, which is the interesting case, when biasing occurs: the spectrum is continuous, see [42, Prop. 3]. Eigenfunctions such that $\lambda\lambda' \neq 0$ satisfy a degenerate orthogonality relation because, from (31),

$$\left(\frac{1}{\bar{h}(v)} - \frac{\lambda v}{\bar{h}(v)}\right)\zeta_\lambda(v)\zeta_{\lambda'}(v) = \zeta_{\lambda'}(v), \qquad \left(\frac{1}{\bar{h}(v)} - \frac{\lambda' v}{\bar{h}(v)}\right)\zeta_{\lambda'}(v)\zeta_\lambda(v) = \zeta_\lambda(v).$$

Subtracting both these expressions, integrating in v and using the normalization yield:

$$\int_{-1}^{1} \frac{v}{\bar{h}(v)}\zeta_\lambda(v)\zeta_{\lambda'}(v)dv = 0 \text{ if } \lambda \neq \lambda'.$$

This property has a surprising consequence: indeed, pick $\lambda' = 0$, $\lambda \neq 0$, it comes,

$$\forall \lambda \neq 0, \qquad \int_{-1}^{1} \frac{v}{\bar{h}(v)}\zeta_\lambda(v)\zeta_0(v)dv = \int_{-1}^{1} v\zeta_\lambda(v)dv = 0,$$

so that only one eigenfunction, associated to $\lambda = 0$, is responsible for the totality of the macroscopic current. The damped modes $\lambda \neq 0$ are endowed with a vanishing moment of order one in v (this was already observed for $\bar{h}(v) = \frac{1}{2}$ in [27, Remark 9.1]). The spectral decomposition holds for $\int_{-1}^{1} v\bar{h}(v)dv \neq 0$ (see also [50]):

$$\bar{f}(x,v) = \alpha\zeta_0(v) + \int_{-1}^{1} A(\lambda)\exp(-\lambda x)\zeta_\lambda(v)\, d\lambda, \qquad \alpha = \frac{\int_{-1}^{1} v\bar{f}(x,v)dv}{\int_{-1}^{1} v\bar{h}(v)dv}. \tag{33}$$

We also mention the work [11], where departing from the similar decomposition as above, the authors show the existence of a stationary solution for a linear kinetic system modelling bacterial chemotaxis by a velocity jump process.

2.2 From stationary Vlasov-Fokker-Planck to Sturm-Liouville

Consider the Fokker-Planck stationary problem with convenient incoming/decay boundaries,

$$v\partial_x\bar{f} = \frac{1}{\theta}\partial_v\big((v-\theta E)\bar{f} + \kappa\partial_v\bar{f}\big), \qquad \theta, \kappa > 0. \tag{34}$$

A convenient "separated variables" ansatz reads now $\bar{f}(x,v) = \exp(-\lambda x - \mu v)\psi_\lambda(v)$, [8], so that one recovers a standard Sturm-Liouville eigenvalue problem, see [27, Chapter 12.3] with a discrete spectrum. The null eigenvalue $\lambda = 0$ is double, its two associated non-damped modes are denoted $\Psi_0^\pm$ ("diffusion solutions" in the sense of [22]) among which appears a space-homogeneous mode:

- when $E > 0$,

$$\Psi_0^+(x,v) = \exp(-\frac{(v-|\theta E|)^2}{2\kappa}); \quad \Psi_0^-(x,v) = \exp(-\frac{|\theta E|^2}{2\kappa})\exp(\frac{Ex}{\kappa} - \frac{v^2}{2\kappa}); \tag{35}$$